Numerical computation of exit times

ANR-09-SEGI-004

Partial report of team CQFD on task 2

This work concerns the evaluation of reliability indices with a stochastic method. Namely, we are focused here on the computation of the law of exit times for piecewise deterministic Markov processes. We propose a numerical scheme to approximate the survival function and moments of a wide class of exit times, analyze the convergence of our scheme and derive bounds for the convergence rate in some cases. The partners involved are INRIA CQFD and Astrium. People involved are A. Brandejsky, B. de Saporta (CQFD), F. Dufour (CQFD) and C. Elegbede (Astrium). This work was not planned in the initial project.

1 Context

Piecewise-deterministic Markov processes (PDMP's) have been introduced in the literature by M.H.A. Davis [9] as a general class of stochastic models. PDMP's are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP $\{X(t)\}$ depends on three local characteristics, namely the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\phi(x,t)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\phi(x,t))$ or when the flow $\phi(x,t)$ hits the boundary of the state-space. In either case the location of the process at the jump time T_1: $X(T_1) = Z_1$ is selected by the transition measure $Q(\phi(x,T_1),\cdot)$. Starting from Z_1, we now select the next interjump time $T_2 - T_1$ and postjump location $X(T_2) = Z_2$. This gives a piecewise deterministic trajectory for $\{X(t)\}$ with jump times $\{T_k\}$ and post jump locations $\{Z_k\}$ which follows the flow ϕ between two jumps. A suitable choice of the state space and the local characteristics ϕ, λ, and Q provides stochastic models covering a great number of problems of operations research [9].

Numerical computation of the moments of the exit time for a Markov process has been studied by K. Helmes, S. Röhl and R.H. Stockbridge in [10]. Starting from an assumption related to the generator of the process, they derive a system of linear equations satisfied by the moments. In addition to these equations, they include finitely many Hausdorff moment conditions that are also linear constraints. This optimization problem is a standard linear programming problem for which many efficient softwares are available. J.-B. Lasserre and T. Prieto-Rumeau introduced in [11] a similar method but they improved the efficiency of the algorithm by replacing the Hausdorff moment conditions with semidefinite positivity constraints of some moment matrices. Nevertheless, their approach cannot be applied to PDMP's because the assumption related to the generator of the process is generally not satisfied. In [9] section 33, M.H.A. Davis gives an iterative method to compute the mean exit time for a PDMP but his approach involves solving a large set of ODEs whose forms are very problem specific, depending on the behaviour of the process at the boundary of the state space. Besides, and in the context of applications to reliability, it seems important to study also the distribution of the exit time.

2 Approach

We consider a PDMP $(X_t)_{t\geq 0}$ with state space E and we present approximation methods to compute the moments and the survival function of the exit time from a set denoted $U \subset E$ given the fact that it happens before the N^{th} jump of the PDMP denoted by T_N. Roughly speaking, we estimate the moments and the survival function for $\tau \wedge T_N$. In our approach, the first step consists in expressing the j-th moment (respectively the survival function) as the last term of some sequence $(p_{k,j})_{k\leq N}$ (respectively $(p_k)_{k\leq N}$) satisfying a recursion $p_{k+1,j} = \psi(p_{k,j})$ (respectively $p_{k+1} = \psi(p_k)$) specifically built within our paper.

In this context, a natural way to deal with these problems is to follow the idea developed in [8] namely to write the recursions in terms of an underlying discrete-time Markov chain and to replace it by its quantized approximation. The definitions of $(p_{k,j})_k$ and $(p_k)_k$ involve some discontinuities related to indicator functions but as in [8], we show that they happen with small enough probability. However, an important feature that distinguishes the present work from [8] and which prevents a straightforward application of the ideas developed within, is that an additional important difficulty appears in the definition of the sequences $(p_{k,j})_k$ and $(p_k)_k$. Indeed, the mapping ψ such that $p_{k+1,j} = \psi(p_{k,j})$ and $p_{k+1} = \psi(p_k)$ is not Lipschitz continuous.

3 Results

We managed to overcome this difficulty by deriving new and important properties of the Markov chain $(Z_n, T_n)_{n\in\mathbb{N}}$, combined to a sharp feature of the quantization algorithm. We proposed quantization-based approximation schemes for the survival function and moments of the exit time. We are able to prove the convergence of the approximation scheme. Moreover, in the case of the moments, we even obtain bounds for the rate of convergence. It is important to stress that these assumptions are quite reasonable with regards to the applications.

The main practical interests of our approach are the following.

- The quantizations grids only have to be computed once and for all and can be used for several purposes. Moreover, once they are obtained, the procedures leading to $\widehat{p}_N(s)$ and to $\widehat{p}_{N,j}$ can be achieved very simply since we only have to compute finite sums.
- Concerning the distribution, since $\widehat{p}_N(s)$ can be computed almost instantly for any value of s, the whole survival function can be obtained very quickly. Similarly, concerning the moments, $\widehat{p}_{N,j}$ can be computed very quickly for any j, so that any moment is almost instantly available.
- Furthermore, in both cases, one may decide to change the set U and consider the exit time τ' from a new set U'. This will yield new sequences $(\widehat{q}_k)_k$, $(\widehat{r}_{k,j})_k$ and $(\widehat{p}_{k,j})_k$ in the case of the j-th moment approximation or new sequences $(\widehat{q}_k)_k$, $(\widehat{r}_k(s))_k$ and $(\widehat{p}_k(s))_k$ if we are interested in the distribution. These new sequences are obtained quickly and easily since the quantized process remains the same and we only have to compute finite sums. Of course, the set U' must be such that Assumptions **??** to **??** remain true and such that $\mathbf{P}_\mu(T_N < \tau')$ remains small without changing the computation horizon N. This last condition is fulfilled if, for instance, $U' \subset U$. This flexibility is an important advantage of our method over, for instance, a Monte Carlo method.

4 Applications

The numerical procedure described above has been first tested on a simple academic model of PDMP derived from a Poisson process. Then, we have successfully implemented it to compute the distribution and moments of an exit time for a corrosion problem proposed by Astrium.

5 Dissemination of results

The theoretical part of this work with rigorous proofs and the academic example is to appear in an international peer-reviewed journal *Advances in Applied Probability* [1]. The detailed study of the corrosion model proposed by Astrium is given in the proceeding of *ESREL* conference, a European peer-reviewed conference on reliability and safety, and co-signed by CQFD and Astrium [2]. A. Brandejsky and F. Dufour presented this work in several national and international seminars and conferences [7, 6, 5, 3, 4].

References

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