Numerical method for the impulse control of Piecewise Deterministic Markov Processes

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Partial report of team CQFD and Astrium on task 3.1.3

The aim of this task is to propose a an application of our numerical optimal stopping procedure to an example of corrosion in collaboration with Astrium. The partners involved are INRIA CQFD. People involved are B. de Saporta (CQFD) and F. Dufour (CQFD) with the collaboration of H. Zhang (CQFD).

1 Context

Piecewise-deterministic Markov processes (PDMP's) have been introduced in the literature by [6] as a general class of stochastic hybrid models. PDMP's are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP includes both continuous and discrete variables $\{(X(t), \Upsilon(t))\}$. The hybrid state space (continuous/discrete) is defined as $\mathbb{R}^d \times M$ where M is a countable set. The process depends on three local characteristics, namely the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from $(x, \nu) \in \mathbb{R}^d \times M$ the motion of the process follows the trajectory $(\phi_\nu(x, t), \nu)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda_\nu(\phi_\nu(x, t))$ or when the flow $\phi_\nu(x, t)$ hits the boundary of the state-space. In either case the location of the process at the jump time T_1: $(X(T_1), \Upsilon(T_1)) = (Z_1, y_1)$ is selected by the transition measure $Q_\nu(\phi_\nu(x, T_1), \cdot)$. Starting from (Z_1, y_1), we now select the next inter-jump time $T_2 - T_1$ and postjump location $(X(T_2), \Upsilon(T_2)) = (Z_2, y_2)$. This gives a piecewise deterministic trajectory for $\{(X(t), \Upsilon(t))\}$ with jump times $\{T_k\}$ and post jump locations $\{(Z_k, y_k)\}$ which follows the flow ϕ between two jumps. A suitable choice of the state space and the local characteristics ϕ, λ, and Q provides stochastic models covering a great number of problems of operations research, see [6]. To simplify notation, there is no loss of generality in considering that the state space of the PDMP is taken simply as a subset of $\mathbb{R}^d$ rather than a product space $\mathbb{R}^d \times M$ as described above, see Remark 24.9 in [6] for details.

An impulse control strategy consists in a sequence of single interventions introducing a jump of the process at some controller-specified stopping time and moving the process at that time to some new point in the state space. Our impulse control problem consists in choosing a strategy (if it exists) that minimizes the expected sum of discounted running and intervention costs up to infinity, and computing the optimal cost thus achieved. Many applied problems fall into this class, such as inventory problems in which a sequence of restocking decisions is made, or optimal maintenance of complex systems with components subject to failure and repair.

Impulse control problems of PDMP's in the context of an expected discounted cost have been considered in [5, 9, 10, 11, 14]. Roughly speaking, in [5] the authors study this impulse control problem by using the value improvement approach while in [9, 10, 11, 14] the authors choose to analyze it by using the variational inequality approach. In [5], the authors also consider a numerical procedure. By showing that iteration of the single-jump-or-intervention operator generates a sequence of functions converging to the value function of the problem, they derive an algorithm to compute an approximation of that value function. Their approach is also based on a uniform discretization of the state space similar to the one proposed by [13]. In particular, they derive a convergence result for the approximation scheme but no estimation of the rate of convergence is given. To the best of our knowledge, it is the only paper presenting a computational method for solving the impulse control problem for a PDMP in the context of discounted cost. Remark that a similar procedure has been applied by [3] to derive a numerical scheme for the impulse control problem with a long run average cost.

2 Approach

Our approach is also based on the iteration of the single-jump-or-intervention operator, but we want to derive a convergence rate for our approximation. Our method does not rely on a *blind* discretization of the state space, but on a discretization that depends on time and takes into account the random nature of the process. Our approach involves a quantization procedure. Roughly speaking, quantization is a technique that approximates a continuous state space random variable X by a a random variable $\widehat{X}$ taking only finitely many values and such that the difference between X and $\widehat{X}$ is minimal for the L_p norm. Quantization methods have been developed recently in numerical probability, nonlinear filtering or optimal stochastic control with applications in finance, see e.g. [1, 2, 15, 16, 17, 18] and references therein. It has also been successfully used by the authors to compute an approximation of the value function and optimal strategy for the optimal stopping problem for PDMP's in [de Saporta et al.(2010)de Saporta, Dufour, and Gonzalez].

Although the value function of the impulse control problem can be computed by iterating implicit optimal stopping problems, see [5] Proposition 2 or [6] Proposition 54.18, from a numerical point of view the impulse control is much more difficult to handle than the optimal stopping problem. Indeed, for the optimal stopping problem, the value function is computed as the limit of a sequence (v_n) constructed by iterating an operator L. This iteration procedure yields an iterative construction of a sequence of random variables $v_n(Z_n)$ (where (Z_n) is an embedded discrete-time process). This was the keystone of our approximation procedure. As regards impulse control, the iterative construction for the corresponding random variables does not hold anymore, see Section **??** for details. This is mostly due to the fact that not only does the controller choose times to stop the process, but they also choose a new starting point for the process to restart from after each intervention. This makes the single-jump-or-intervention operator significantly more complicated to iterate that the single-jump-or-stop operator used for optimal stopping. We manage to overcome this extra difficulty by using two series of quantization grids instead of just the one we used for optimal stopping.

3 Results

We have managed to propose a numerical scheme to approximate the value function of a general impulse control problem for PDMPs. We also derived an error bound for the convergence of our scheme. It is based on two different series if quantization grids.

4 Dissemination of results

This work was presented in a conference [8] and published in an international peer-reviewed journal [7].

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