Implementation an optimal stopping algorithm for PDMP's

ANR-09-SEGI-004

Final report of team CQFD on task 3.1.2

The aim of this task is to implement the computational method for optimal stopping of piecewise deterministic Markov processes (PDMP's) designed in [2, 3], develop a matlab code and test it on simple PDMP's. The partner involved is INRIA CQFD. People involved are B. de Saporta (CQFD), F. Dufour (CQFD), with the collaboration of H. Zhang (CQFD).

1 Context

Piecewise-deterministic Markov processes (PDMP's) have been introduced in the literature by M.H.A. Davis [7] as a general class of stochastic models. PDMP's are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP $\{X(t)\}$ depends on three local characteristics, namely the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\phi(x,t)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\phi(x,t))$ or when the flow $\phi(x,t)$ hits the boundary of the state-space. In either case the location of the process at the jump time T_1: $X(T_1) = Z_1$ is selected by the transition measure $Q(\phi(x,T_1),\cdot)$. Starting from Z_1, we now select the next interjump time $T_2 - T_1$ and postjump location $X(T_2) = Z_2$. This gives a piecewise deterministic trajectory for $\{X(t)\}$ with jump times $\{T_k\}$ and post jump locations $\{Z_k\}$ which follows the flow ϕ between two jumps. A suitable choice of the state space and the local characteristics ϕ, λ, and Q provides stochastic models covering a great number of problems of operations research [7].

The optimal stopping problem consists in finding the best time to stop the process in order to optimize the expectation of a reward function g of the process at that time. The best possible performance is called the value function of the problem, and a stopping time reaching this optimal performance is called an optimal stopping time.

An algorithm to compute an approximation of the value function as well as an approximation of an ϵ-stopping time has been designed by the team INRIA CQFD [2] and presented in a previous report [3]. We present here the results obtained on a simple PDMP example.

2 Example

Our simple PDMP is as follows. The state space is $E = [0,1[$, and i ts boundary is $\partial E = \{1\}$. The flow is defined on $[0,1]$ by $\phi(x,t) = x + vt$ for some positive v, the jump rate is defined on $[0,1]$ by $\lambda(x) = \beta x^\alpha$, with $\beta > 0$ and $\alpha \geq 1$, and for all $x \in [0,1]$, one sets $Q(x,\cdot)$ to be the uniform law on $[0,1/2]$. Thus, the process moves with constant speed v towards 1, but the closer it gets to the boundary 1, the higher the probability to jump backwards on $[0,1/2]$. Figure 1 shows two trajectories of this process for $x_0 = 0$, $v = \alpha = 1$ and $\beta = 3$ and up to the 10-th jump.

The reward function g is defined on $[0,1]$ by $g(x) = x$. The assumptions of validity of our approximation procedure all hold for this simple process. All the constants involved in the derivation of the convergence rates can be computed explicitly.

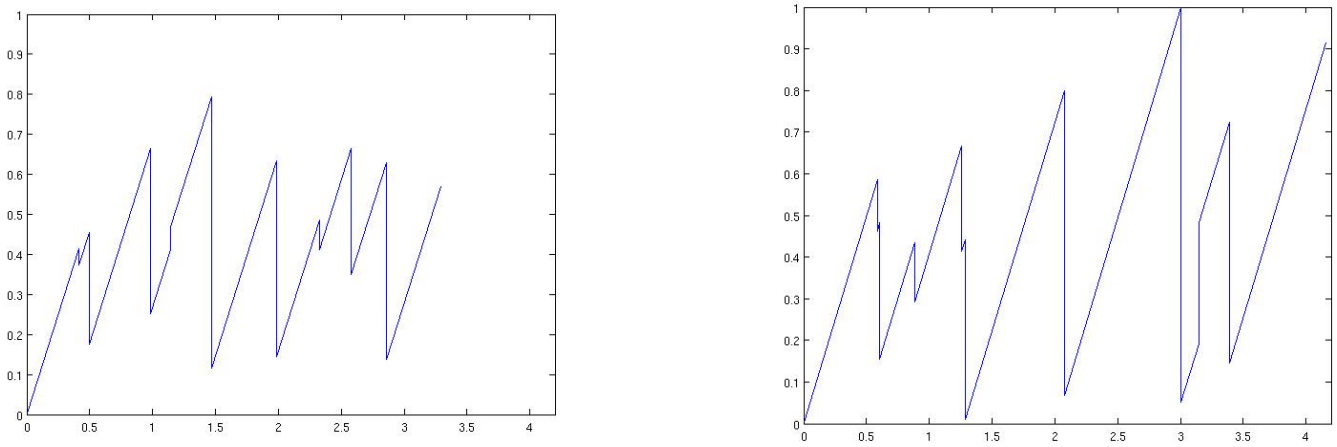

Figure 1: Two trajectories of the PDMP.

3 Results

The real value function $V_0 = v_0(x_0)$ is unknown, but, as our stopping rule τ_N is a stopping time dominated by T_N, one clearly has

$$\overline{V}_0 = \mathbf{E}_{x_0}\big[g(X(\tau_N))\big] \leq V_0 = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_{x_0}\big[g(X(\tau))\big] \leq \mathbb{E}_{x_0}\big[\sup_{0 \leq t \leq T_N} g(X(t))\big].$$

The first and last terms can be evaluated by Monte Carlo simulations, which provides another indicator of the sharpness of our numerical procedure. For 10^6 Monte Carlo simulations, one obtains $\mathbb{E}_{x_0}\big[\sup_{0 \leq t \leq T_N} g(X(t))\big] = 0.9878$. Simulation results (for $d = 2$, $x_0 = 0$, $v = \alpha = 1$, $\beta = 3$, up to the 10-th jump and for 10^5 Monte Carlo simulations) are given in Table 1. Note that, as expected, the theoretical errors decrease as the quantization error decreases. One also has

$$V_0 - \overline{V}_0 \leq \mathbb{E}_{x_0}\big[\sup_{0 \leq t \leq T_N} g(X(t))\big] - \overline{V}_0.$$

This provides an empirical upper bound for the error.

4 Dissemination of results

The theoretical part of this work with rigorous proofs and the academic example is published in an international peer-reviewed journal *Annals of Applied Probability* [2]. The detailed study of the corrosion model proposed by Astrium is given in the proceeding of *Lambda-Mu 17* conference, a French peer-reviewed conference on reliability and safety, and co-signed by CQFD and Astrium [4]. It is also accepted for publication in an international peer-reviewed journal [5]. B. de Saporta and F. Dufour were invited to present these results at the workshop *Modern trends in controlled stochastic processes* in Liverpool [1]. F. Dufour was invited to present these results at SPA2010 conference in Osaka [6].

Pt	QE	Δ	$\widehat{V}_0$	$\overline{V}_0$	B_1	B_2	B_3
10	0.0943	0.151	0.7760	0.8173	0.1705	74.64	897.0
50	0.0418	0.100	0.8298	0.8785	0.1093	43.36	511.5
100	0.0289	0.083	0.8242	0.8850	0.1028	34.15	400.3
500	0.0133	0.056	0.8432	0.8899	0.0989	21.03	243.1
900	0.0102	0.049	0.8514	0.8968	0.0910	17.98	206.9

Pt	Number of points in each quantization grid
QE	Quantization error: $QE = \max_{0\leq k\leq N} \|\Theta_k - \widehat{\Theta}_k\|_2$
Δ	For all z, $\Delta(z) = \Delta$
B_1	Empirical bound $\mathbb{E}_{x_0}[\sup_{0\leq t\leq T_N} g(X(t))] - \overline{V}_0$
B_2	Theoretical bound for $\|\widehat{V}_0 - V_0\|$
B_3	Theoretical bound for $\|\overline{V}_0 - V_0\|$

Table 1: Simulation results.

References

Publications of the Fautocoes team

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[3] de Saporta, B., Dufour, F., Elegbede, C., Gonzalez and K., Zhang, H. Analysis of convergence and derivation of a rate of convergence for optimal stopping, avec F. Dufour, Rapport de CQFD pour l'ANR 2011

[4] de Saporta, B., Dufour, F., Zhang, H. and Elegbede, C. Arrêt optimal pour la maintenance prédictive In *Lambda-mu 17* (La Rochelle, 2010).

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References in the text

[7] Davis, M. H. A. *Markov models and optimization*, vol. 49 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London, 1993.